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Fourier transforms of distributions and associated Feynman integrals

H P Berg and Jan Tarski

Institut für Theoretische Physik, Technische Universität Clausthal, D-3392 Clausthal-Zellerfeld, Federal Republic of Germany

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Abstract. Infinite-dimensional distributions and their Fourier transforms are considered. Some of such transforms are shown to be Feynman-integrable. From the resulting Feynman integrals an equation of motion and a weak form of canonical commutation relations are obtained.

1. Introduction

Itô observed (1966) that Fourier transforms of bounded measures can be very effectively handled in Feynman integration. In particular, Feynman integrals of these functions can be reduced to measure-theoretic integrals, and this circumstance allows one to perform (with rigour) various useful manipulations. These functions and their integrals were subsequently investigated extensively by Albeverio and Høegh-Krohn, and to a lesser extent by Chebotarev and Maslov, Truman, and others. (See e.g. the articles in Albeverio *et al* (1979, § I) and references cited therein.)

It is natural to extend this line of research to Feynman integrals of Fourier transforms of distributions. The infinite-dimensional distributions are of course not as well understood as finite-dimensional ones, cf Krée (1976). However, at least some of such distributions are derivatives of bounded measures, and then their Fourier transforms differ from Fourier transforms of measures by polynomial factors. The corresponding Feynman integrals therefore reduce to integrals of known integrands with respect to the given measures.

In this article we confine ourselves to a rather special class of distributions. This class suffices to establish equations of motion and a form of canonical commutation relations. We recall that these applications were discussed heuristically on various occasions (Feynman 1948, Feynman and Hibbs 1965). For the present (rigorous) analysis we assume that the potentials are Fourier transforms of measures satisfying certain boundedness properties. The equations that we establish are of course expected to have a much broader validity.

We note also the following. Since the Feynman integrals in question can be done in closed form, it is largely immaterial which definition of these integrals is adopted. For definiteness we will employ the two definitions given in Tarski (1979, 1980).

2. Feynman integrability

To orient ourselves, let us first consider a one-dimensional integral, and the case where the distribution is the first derivative of a measure λ. We write the Fourier transform as

$$f(u)=\int [\partial_x \,\mathrm{d}\lambda(x)]\,\mathrm{e}^{\mathrm{i}xu} = -\int \mathrm{d}\lambda(x)\partial_x \,\mathrm{e}^{\mathrm{i}xu} = -\mathrm{i}u\int \mathrm{d}\lambda(x)\,\mathrm{e}^{\mathrm{i}xu}. \tag{1}$$

A formal evaluation of the Feynman integral of f proceeds as follows:

$$\begin{aligned}(-\mathrm{i}\kappa/2\pi)^{1/2}&\int \mathrm{d}u\,\exp(\mathrm{i}\kappa u^2/2)f(u)\\ &= -\int \mathrm{d}\lambda(x)\,\partial_x\Big((-\mathrm{i}\kappa/2\pi)^{1/2}\int \mathrm{d}u\,\exp(\mathrm{i}\kappa u^2/2)\,\mathrm{e}^{\mathrm{i}xu}\Big)\\ &= -\int \mathrm{d}\lambda(x)\,\partial_x\,\exp(x^2/2\mathrm{i}\kappa) = -(\mathrm{i}\kappa)^{-1}\int \mathrm{d}\lambda(x)\,x\,\exp(x^2/2\mathrm{i}\kappa).\end{aligned} \tag{2}$$

We see that we have to require $\int \mathrm{d}|\lambda|(x)(1+|x|)<\infty$, where $|\lambda|$ is the absolute variation of λ, and the need for the 1 in $(1+|x|)$ is evident in equations (1).

We turn to the infinite-dimensional situation and to mathematical details. We assume a real Hilbert space $\mathscr{H}$, and the Feynman integral with the weight $\exp(\mathrm{i}\kappa\langle\xi,\xi\rangle/2)$, where $\operatorname{Im}\kappa \geqslant 0$, $\kappa\neq 0$. In this article Feynman integrability refers both to the definition in terms of projections and to the definition in the sense of analytic continuation (Tarski 1979, 1980 respectively). For the former definition, the reference family of sequences of projections is the maximal family $\hat{\mathscr{Q}}$ (but cf the subsequent qualification).

We denote by D_ζ the derivative associated with the vector $\zeta\in\mathscr{H}$. Since such derivatives will affect only a finite number of coordinates, and will act on polynomials and exponentials, it is irrelevant whether we assume the derivative of Gâteaux or of Fréchet. Next, let μ be a Borel measure on $\mathscr{H}$ of bounded absolute variation, and let $\zeta_1,\ldots,\zeta_k\in\mathscr{H}$. Then $D_{\zeta_1}\ldots D_{\zeta_n}\mu$ is a distribution, whose Fourier transform in analogy to (1) is

$$\begin{aligned}F(\xi)&=\int_{\mathscr{H}}[D_{\zeta_1}\ldots D_{\zeta_n}\,\mathrm{d}\mu(\chi)]\exp(\mathrm{i}\langle\chi,\xi\rangle)\\ &=(-\mathrm{i})^n\langle\zeta_1,\xi\rangle\ldots\langle\zeta_n,\xi\rangle\int \mathrm{d}\mu(\chi)\exp(\mathrm{i}\langle\chi,\xi\rangle).\end{aligned} \tag{3}$$

Proposition 1. The above Fourier transform F is Feynman integrable provided

$$\int \mathrm{d}|\mu|(\chi)(1+|\langle\zeta_1,\chi\rangle|)\ldots(1+|\langle\zeta_n,\chi\rangle|)<\infty, \tag{4a}$$

and under this hypothesis,

$$\int \mathscr{D}(\xi)\exp(\mathrm{i}\kappa\langle\xi,\xi\rangle/2)F(\xi)=(-\mathrm{i})^n\int \mathrm{d}\mu(\chi)\,D_{\zeta_1}\ldots D_{\zeta_n}\exp(\langle\chi,\chi\rangle/2\mathrm{i}\kappa). \tag{4b}$$

Proof. First we consider the definition in the sense of analytic continuation. We replace

$-\mathrm{i}\kappa$ by b' and suppose $b, b' > 0$. We are then interested in the following measure-theoretic integral, which can be done in closed form:

$$J(b, \alpha, b'; F)$$
$$= \int \mathscr{D}(\xi) \exp(-b\langle \xi - \alpha, \xi - \alpha\rangle/2) \exp(-b'\langle \xi, \xi\rangle/2)$$
$$\times (-\mathrm{i})^n \langle \zeta_1, \xi\rangle \ldots \langle \zeta_n, \xi\rangle \int \mathrm{d}\mu(\chi) \exp(\mathrm{i}\langle \chi, \xi\rangle)$$
$$= \exp(-b\langle \alpha, \alpha\rangle/2) \int \mathrm{d}\mu(\chi) \langle \zeta_1, \delta/\delta\chi\rangle \ldots \langle \zeta_n, \delta/\delta\chi\rangle$$
$$\times \exp[-\tfrac{1}{2}(b + b')^{-1}\langle \chi - \mathrm{i}b\alpha, \chi - \mathrm{i}b\alpha\rangle]. \tag{5}$$

(Here the last scalar product is bilinear, or symmetric, even though $\chi - \mathrm{i}b\alpha$ is in the complexified space.) If $\mathrm{Re}(b + b') > 0$, then the integrand is a damped gaussian which is modified by functions of slower growth in a finite number of directions. It follows that J is analytic in b, b' if $\mathrm{Re}(b + b') > 0$, and we may let $b' = -\mathrm{i}\kappa$.

In order to justify passing to the limit $b \to 0$, we recall the bound (Tarski 1979, inequality (3.11); this inequality presupposes $|\mathrm{Im}\, b| \leq \frac{1}{2}|\mathrm{Re}\, \kappa|$):

$$|\exp[-\tfrac{1}{2}(b - \mathrm{i}\kappa)^{-1}\langle \chi, \chi - 2\mathrm{i}b\alpha\rangle] \leq \exp(4|\mathrm{Re}\, \kappa|\langle \alpha, \alpha\rangle). \tag{6}$$

This bound and the hypothesis on μ, together with the bounded convergence theorem, justify taking the limit $b \to 0$ inside the μ-integral.

For the definition in terms of projections, we start with the evaluation (5), where the replacements $b' \to -\mathrm{i}\kappa$, $\chi \to P_j\chi$, $\alpha \to P_j\alpha$ are made. Note that if $\mathrm{Re}\, b > 0$, then we are in the case $\mathrm{Re}\,(b + b') > 0$ considered before, the integrand in (5) contains a damped gaussian and is bounded, and so the integral converges. Furthermore, P_j occurs in effect only in the combinations $P_j\chi$ and $\langle P_j\alpha, P_j\alpha\rangle$. But $\sup_{\chi \in P_j\mathscr{H}} \leq \sup_{\chi \in \mathscr{H}}$, and the factors involving $\langle P_j\alpha, P_j\alpha\rangle$ can be readily majorised independently of P_j. We thus obtain a bound independent of P_j, and the bounded convergence theorem allows us to take $P_j \to 1$ inside the integral. The limit $b \to 0$ then proceeds as before.

For applications we will need some estimates for convolutions of measures. Consider an entire function φ and the associated convolution-series:

$$\varphi(z) = \sum a_j z^j, \qquad \mu_\varphi = \sum a_j \mu^{(*,j)}, \tag{7}$$

where $\mu^{(*,j)} = \mu * \ldots * \mu$ (j times).

Lemma 2. Let $\zeta_1, \ldots, \zeta_n \in \mathscr{H}$, let k be an integer ≥ 0, and let

$$\pi(\chi) = (1 + |\langle \zeta_1, \chi\rangle|) \ldots (1 + |\langle \zeta_n, \chi\rangle|)(1 + \|\chi\|)^k. \tag{8a}$$

Suppose that

$$\int \mathrm{d}|\mu|(\chi)\pi(\chi) = M < \infty, \qquad \int \mathrm{d}|\lambda|(\chi)\pi(\chi) = M_1 < \infty. \tag{8b}$$

Then

$$\int \mathrm{d}|\mu * \lambda|(\chi)\pi(\chi) \leq MM_1, \qquad \int \mathrm{d}|\mu_\varphi|(\chi)\pi(\chi) \leq \sum_j |a_j| M^j < \infty. \tag{8c, d}$$

Proof. We will establish (8*c*), and then (8*d*) will follow by repeating the same kind of argument and by taking the limit $j \to \infty$. Let $\chi = \chi' + \chi''$. Then, in view of the triangle inequality,

$$1 + \|\chi\| \leqslant (1 + \|\chi'\|)(1 + \|\chi''\|),$$

and there is an analogous inequality involving $1 + |\langle \zeta, \chi \rangle|$. Therefore

$$\pi(\chi) \leqslant \pi(\chi')\pi(\chi''), \tag{9}$$

and

$$\int \mathrm{d}|\mu * \lambda|(\chi)\pi(\chi) \leqslant \int \mathrm{d}|\mu|(\chi')\, \mathrm{d}|\lambda|(\chi - \chi')\pi(\chi)$$
$$\leqslant \int \mathrm{d}|\mu|(\chi') \int \mathrm{d}|\lambda|(\chi'')\pi(\chi')\pi(\chi''). \tag{10}$$

3. An equation of motion

We consider a quantum particle on R^n, evolving during the time interval $[0, t]$. The associated Hilbert space $\mathscr{H}$ of paths is then determined by

$$\langle \dot\eta, \dot\eta \rangle = \int_0^t \mathrm{d}\tau \sum_{j=1}^n (\mathrm{d}\eta^j/\mathrm{d}\tau)^2, \qquad \eta(0) = 0. \tag{11}$$

We recall that $\eta(T)$ $(0 \leqslant T \leqslant t)$ is a kind of partial scalar product,

$$\eta(T) = \int_0^t \mathrm{d}\tau\, \dot\beta_T(\tau)\dot\eta(\tau) \qquad \text{where } \beta_T(\tau) = \min(T, \tau). \tag{12}$$

If the integrand contains the factor $\delta(\eta(t) - y)$ corresponding to an end-point condition, this can be handled conveniently by introducing the projection P onto the space $\{u\beta_t\colon u \in R^n\}$. Then we may decompose

$$\eta(\tau) = \eta_0(\tau) + \tau\eta(t)/t, \qquad \eta_0(\cdot) \in (1-P)\mathscr{H} =: \mathscr{H}_0, \qquad (\cdot)\eta(t)/t \in P\mathscr{H}. \tag{13}$$

We also set $\beta_{T,0}(\tau) = \beta_T(\tau) - \tau T/t$.

Our first concern is to investigate the conditions on the potential. We recall the observation of Itô (1966), that a potential which is the Fourier transform of a bounded measure on R^n yields an integrand for the Feynman integral which is a similar transform over $\mathscr{H}_0$. We write therefore

$$V(\boldsymbol{x}) = \int_{R^n} \mathrm{d}\nu(\boldsymbol{u})\, \mathrm{e}^{\mathrm{i}\boldsymbol{u}\boldsymbol{x}}, \tag{14a}$$

and we will consider conditions of the form (where k is an integer $\geqslant 0$)

$$\int \mathrm{d}|\nu|(\boldsymbol{u})(1 + |\boldsymbol{u}|)^k =: p_k < \infty. \tag{14b}$$

We suppose for definiteness that the integrand contains the factor $\delta(\eta(t)-y)$. Then

$$-\mathrm{i}\int_0^t \mathrm{d}\tau\, V[\eta_0(\tau)+y\tau/t] = -\mathrm{i}\int_0^t \mathrm{d}\tau \int \mathrm{d}\nu(\boldsymbol{u}) \exp(\mathrm{i}\langle \boldsymbol{u}\dot{\beta}_{\tau,0}, \dot{\eta}_0\rangle) \exp(\mathrm{i}\boldsymbol{u}\boldsymbol{y}\tau/t)$$

$$=: \int_{\mathscr{H}_0} \mathrm{d}\mu_1(\chi) \exp(\mathrm{i}\langle\dot{\chi}, \dot{\eta}_0\rangle), \tag{15}$$

where the factor $\exp(\mathrm{i}\boldsymbol{u}\boldsymbol{y}\tau/t)$ has been absorbed into μ_1, and where χ ranges over the set $\{\boldsymbol{u}\beta_{\tau,0}\}$. Since

$$\|\boldsymbol{u}\dot{\beta}_{\tau,0}\| \leq \|\boldsymbol{u}\dot{\beta}_\tau\| = \tau^{1/2}|\boldsymbol{u}| \leq t^{1/2}|\boldsymbol{u}|, \tag{16}$$

we have $\|\dot{\chi}\| \leq t^{1/2}|\boldsymbol{u}|$. We use, moreover, the following inequality, which can be easily verified by induction and differentiation:

$$(1+c_1c_2)^k \leq (1+c_1)^k(1+c_2)^k \qquad \text{for } c_1, c_2 \geq 0, \tag{17}$$

and obtain (since $|\mu_1 \exp(\mathrm{i}\boldsymbol{k}\boldsymbol{y})| \leq 2|\mu_1|$)

$$\int \mathrm{d}|\mu_1|(\chi)(1+\|\dot{\chi}\|)^k \leq 2\int \mathrm{d}\tau\, \mathrm{d}|\nu|(\boldsymbol{u})(1+t^{1/2}|\boldsymbol{u}|)^k \leq 2t(1+t^{1/2})^k p_k. \tag{18}$$

The foregoing analysis and lemma 2 now yield:

Lemma 3. If V is such that a certain $p_k<\infty$, and $\varphi(z)=\Sigma\, a_j z^j$ is entire, then

$$\varphi\left(-\mathrm{i}\int_0^t \mathrm{d}\tau\, V[\eta_0(\tau)+y\tau/t]\right) = \int_{\mathscr{H}_0} \mathrm{d}\mu_\varphi(\chi) \exp(\mathrm{i}\langle\dot{\chi}, \dot{\eta}_0\rangle), \tag{19a}$$

where μ_φ is determined by the convolution series of μ_1 (cf (15) and (7)), and satisfies

$$\int_{\mathscr{H}_0} \mathrm{d}|\mu_\varphi|(\chi)(1+\|\dot{\chi}\|)^k \leq \sum_j |a_j|[2t(1+t^{1/2})^k p_k]^j < \infty. \tag{19b}$$

We will use this lemma with $\varphi(z)=\mathrm{e}^z$ and also with $\varphi(z)=\mathrm{e}^z-1$. With this lemma we are prepared for:

Proposition 4. Let V be such that $p_1<\infty$. Let $f\in\mathscr{H}_0$ be continuously differentiable and vanishing near 0 and t. Then the two terms of the following integrand are integrable, and one has the equality

$$\int_{\eta(0)=0} \mathscr{D}(\eta) \exp(\mathrm{i}m\langle\dot{\eta}, \dot{\eta}\rangle/2) \exp\left(-\mathrm{i}\int_0^t \mathrm{d}\tau\, V[\eta(\tau)]\right)\delta(\eta(t)-y)$$

$$\times \int_0^t \mathrm{d}\sigma\, f(\sigma)[m\ddot{\eta}(\sigma)+\nabla V[\eta(\sigma)]] = 0. \tag{20}$$

The function $\ddot{\eta}$ is to be interpreted in the sense of distributions: $\langle f, \ddot{\eta}\rangle = -\langle\dot{f}, \dot{\eta}\rangle$.

The Feynman integral may be considered as over $\mathscr{H}_0$ (after the δ-function is integrated out), or over $\mathscr{H}$. In the latter case one should use approximations defined by projections $P_j \geq P$ (i.e. one should use the family $\mathscr{Q}(P)$ of sequences, cf Tarski (1979)).

Proof. The equality was proved in Tarski (1976) under the assumption of integrability.

That article was based on a different definition of the integral, but the proof depends only on the integration-by-parts formula, which remains valid in the present case. (See Tarski 1979, § 4.)

Now, the integrability of the term involving $\langle f, \ddot{\eta}\rangle = -\langle \dot{f}, \dot{\eta}\rangle$ follows from proposition 1: Since $1+|\langle \dot{f}, \dot{\eta}\rangle| \leq (1+\|\dot{f}\|)(1+\|\dot{\eta}\|)$, lemma 3 with $\varphi(z)=\mathrm{e}^z$ ensures that the hypothesis of proposition 1 is fulfilled. Furthermore, $\partial V/\partial x^j$ is the Fourier transform of $\mathrm{i}u^j\nu(u)$, which is a bounded measure (since $p_1<\infty$). Then $\int \mathrm{d}\sigma f \nabla V$ is the Fourier transform of a bounded measure μ_f on $\mathscr{H}_0$, and the second term as a whole is the Fourier transform of $\mu_f * \mu_\varphi$, and so is integrable in view of lemma 2.

4. Canonical commutation relations

We consider again time evolution of a quantum particle. Heuristically, the canonical commutation relations take the following form. Insert

$$m\dot{\eta}^j(T+\varepsilon)\eta^k(T)-\eta^k(T)m\dot{\eta}^j(T-\varepsilon)-\mathrm{i}^{-1}\delta^{jk}$$

as a factor in the integrand for the Green function, and then the integrand should $\to 0$ as $\varepsilon \searrow 0$. We offer a partial result toward establishing this limit relation. For brevity in writing we now assume a particle on R^1.

Lemma 5. Let V be such that $p_2<\infty$. Take $[0, s]$ as the time interval, suppose $0<T-\Delta<T+\Delta<s$, and let $f\in\mathscr{H}_0$ be such that $\dot{f}(\tau)=0$ for $\tau>\Delta$. Consider the following integral:

$$K(s;y,x)=\int_{\eta(0)=0}\mathscr{D}(\eta)\exp(\mathrm{i}m\langle\dot{\eta},\dot{\eta}\rangle/2)\exp\left(-\mathrm{i}\int_0^s \mathrm{d}\tau\, V[\eta(\tau)+x]\right)\delta(\eta(s)+x-y)$$
$$\times\int_0^\Delta \mathrm{d}\sigma\,\dot{f}(\sigma)\{m\dot{\eta}(T+\sigma)[\eta(T)+x]-[\eta(T)+x]m\dot{\eta}(T-\sigma)-\mathrm{i}^{-1}\}. \tag{21}$$

Then this Feynman integral converges. Furthermore, if $s\searrow 0$ and $\|\dot{f}\|$ remains constant, then $K\sim \mathrm{O}(s)$. Explicitly,

$$|K|\leq(|x|+|y|+3s^{1/2})(1+m^{-1/2})^2\|\dot{f}\|\{\exp[2s(1+s^{1/2})^2p_2]-1\}. \tag{22}$$

Observe the following. (a) The above path integral should be regarded as an integral over $\mathscr{H}_0$, or else one should use projections $P_j \geq P$, cf proposition 4. (b) In the notation $K(s;y,x)$ we suppressed some dependences of K. (c) In order to allow for propagation from x to y, and not only from 0 to y, we replaced $\eta(\tau)$ by $\eta(\tau)+x$ in the integrand. (d) The definition (14b) of the p_k presupposes a unit of scale for length, and for mass (since $\hbar=1$).

Outline of proof. Convergence of K, or integrability of the integrand, follows in the same way as the integrability of the term involving $\langle f, \ddot{\eta}\rangle$ in proposition 4. Next, we write $\exp(-\mathrm{i}\int \mathrm{d}\tau\, V)$ as $1+(\mathrm{e}^{\cdot\cdot}-1)$. The first term here corresponds to $V=0$, and it yields an integral whose evaluation is elementary, and which equals zero independently of f. For the other term, we use lemma 3 with $\mathrm{e}^{\cdot\cdot}-1$ as the entire function φ, and we evaluate the integral as in equation (4b). We then majorise this evaluation with the help of (17) and of Schwarz's inequality in such a way as to be able to apply (19*b*). The coefficient $s^{1/2}$ comes from the norm $\|\dot{\beta}_T\|$, as in (16).

This lemma might be useful for establishing the commutation relations in the form that we mentioned in the beginning, in the following way. Start with the composition law, where G is the Green function:

$$K(t; y, x) = \int du\, dv\, G(t-(T+\Delta); y, u)K(2\Delta; u, v)G(T-\Delta; v, x). \tag{23}$$

We would like to prove that $K(t; y, x) \to 0$ as $\Delta \searrow 0$, while lemma 5 shows that $K(2\Delta; u, v) \to 0$. There remains the problem of justifying the interchange of this limit with integration over u and v, and this problem is beyond the scope of the present article.

We conclude with the following remark. In lemma 5 we did not consider the dependence of f on s. Clearly, as $s \searrow 0$, f must also change. We should therefore like to point out that the decrease $K \sim O(s)$ for $\|\dot{f}\| = \text{constant}$ allows us to take f such that $\|\dot{f}\| \sim s^{-1/2}$ as $s \searrow 0$. (Consider e.g. $\dot{f}(\tau) = 3s^{-1}$ for $\frac{1}{3}s < \tau < \frac{2}{3}s$, $\dot{f} = 0$ otherwise. Then $\|\dot{f}\| = (3/s)^{1/2}$, and as $s \searrow 0$, f approaches a step function, and $\dot{f}$, a δ-function. The form of the canonical commutation relations mentioned in the beginning of this section indeed suggests that the δ-function might be admissible here.

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References

Albeverio S, Combe Ph, Høegh-Krohn R, Rideau G, Sirugue-Collin M, Sirugue M and Stora R (ed.) 1979 *Feynman path integrals—Proc. Marseille* 1978 (Berlin: Springer) Lecture notes in physics 106

Feynman R P 1948 *Rev. Mod. Phys.* **20** 367

Feynman R P and Hibbs A R 1965 *Quantum mechanics and path integrals* (New York: McGraw-Hill) especially ch 7

Itô K 1966 in *Proc. 5th Berkeley symp. mathematical statistics and probability* vol II, part I (Berkeley: University of California Press) p 145

Krée P 1976 in *Séminaire Pierre Lelong (analyse) année* 1974/75, ed. P Lelong (Berlin: Springer) Lecture notes in mathematics 524 p 163

Tarski J 1976 in *Complex analysis and its applications* vol III (Vienna: IAEA) p 193

—— 1979 in *Feynman path integrals—Proc. Marseille* 1978 (Berlin: Springer) Lecture notes in physics 106 p 254

—— 1980 in *Functional integration: theory and applications* (*Proc. Louvain-la-Neuve* 1979) ed J P Antoine and E Tirapegui (New York: Plenum) p 143